Soft modes and the counter-rotating terms of the Dicke model

1977 J. Phys. A: Math. Gen. 10 L179

(http://iopscience.iop.org/0305-4470/10/11/001)

J. Phys. A: Math. Gen., Vol. 10, No. 11, 1977. Printed in Great Britain. © 1977

LETTER TO THE EDITOR

Soft modes and the counter-rotating terms of the Dicke model

B V Thompson

The University of Manchester Institute of Science and Technology, PO Box 88, Manchester M 60 1 QD, UK

Received 22 July 1977

Abstract. The presence of counter-rotating terms in the extended Dicke Hamiltonian alters the temperature dependence of the soft-mode frequencies near the critical temperature. At resonance and in the long-wave limit, frequency doubling does not occur. The zero-temperature limit of excitation energies close to equilibrium is satisfactory.

Several authors have discussed the mode softening behaviour in the Dicke model as the system is cooled towards the phase transition temperature. Sadreev and Slavinskii (1976) maintain the field in a coherent state and consider fluctuations in the atomic variables alone. The work of Gilmore (1977) applies to the region above the critical temperature where fluctuations in the atomic inversion operators σ_3 do not affect the results. It would be satisfactory to have a calculation for 'temperature-dependent' energy levels of the model which could be checked against the established ones in the $T=0$ limit. There is too the question of the sensitivity of the critical behaviour to details of the interaction terms of the Hamiltonian. Mode softening goes as $(T-T_c)^{1/2}$ in certain ferroelectrics (Cochran 1973) in contrast to the linear dependence found for the Dicke model in the rotating wave approximation (RWA).

The first step is to adapt Bogolubov's method (Gilmore and Bowden 1976) to the extended Dicke model for which the Hamiltonian is (Thompson 1977):

$$H=\omega_k a_k^\dagger a_k+\sum_l \omega_0\sigma_3(l)+2g\sigma_1(l)(a_k\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{l}}+a_k^\dagger\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}.\boldsymbol{l}}) \tag{1}$$

where $[\sigma_1,\sigma_2]=\mathrm{i}\sigma_3$ etc, while all other symbols have their normal meaning. The thermodynamically equivalent Hamiltonian may be written

$$H_{\mathrm{L}}=H_0+H_1+H_2 \tag{2}$$

$$H_1=\omega_k a_k^\dagger a_k+2g\sum_l \nu_l(a_k\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{l}}+a_k^\dagger\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}.\boldsymbol{l}}) \tag{3}$$

$$H_2=\sum_l \omega_0\sigma_3(l)+2g(\mu\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{l}}+\mu^*\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}.\boldsymbol{l}})\sigma_1(l). \tag{4}$$

We shall not need the c-number H_0 in this article. For brevity, one mode $\boldsymbol{k}$ is considered and we take μ to be real. Since H_1 and H_2 commute, it is easy to compute thermal averages with H_L. The parameters μ, ν_l are chosen so that

$$\langle a_k\rangle_\beta = \mu, \qquad \langle\sigma_1(l)\rangle_\beta = \nu_l \tag{5}$$

and it is well known that this choice minimises the free energy $-\beta^{-1}\ln \mathrm{Tr}\, \mathrm{e}^{-\beta H_L}$. We find

$$\omega_k\mu = -2g\sum_l \nu_l\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}.\boldsymbol{l}} \tag{6}$$

$$\omega_0\nu_l = 4gs_l\mu \cos \boldsymbol{k}\,.\,\boldsymbol{l} \tag{7}$$

where

$$s_l \equiv \langle\sigma_3(l)\rangle_\beta = -\tfrac{1}{2}\cos\theta_l \tanh(\tfrac{1}{2}\beta\omega_0 \sec\theta_l) \tag{8}$$

and

$$\tan\theta_l = 4g\mu\omega_0^{-1}\cos \boldsymbol{k}\,.\,\boldsymbol{l}. \tag{9}$$

Each angle θ_l measures the rotation about the two-axis needed to diagonalise the summand labelled by $\boldsymbol{l}$ in (4). After rejecting the solutions $\mu = 0 = \nu_l$ which describe the disordered state, we obtain

$$1 = 4g^2\omega_0^{-1}\omega_k^{-1}\sum_l \cos^2 \boldsymbol{k}\,.\,\boldsymbol{l}\cos\theta_l \tanh(\tfrac{1}{2}\beta\omega_0\sec\theta_l) \tag{10}$$

and the critical temperature is given by

$$1 = 4g^2\omega_0^{-1}\omega_k^{-1}\tanh(\tfrac{1}{2}\beta_c\omega_0)\sum_l \cos^2 \boldsymbol{k}\,.\,\boldsymbol{l}. \tag{11}$$

At zero temperature, equations (6)–(10) reduce to those found in the canonical transformation theory. The last result we require is that because of the simple structure of H_2,

$$\langle\sigma_2(l)\rangle_\beta = 0.$$

Close to thermal equilibrium, we can introduce fluctuation operators:

$$f \equiv a_k - \mu; \qquad f_1(l) \equiv \sigma_1(l) - \nu_l; \qquad f_3(l) \equiv \sigma_3(l) - s_l.$$

Solution of the linearised equations of motion for these leads to the desired 'temperature-dependent' mode frequencies. From (1), Heisenberg's equations give (ignoring non-linear terms in the fluctuations)

$$\mathrm{i}\dot f = \omega_k f + 2g\sum_l f_1(l)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}.\boldsymbol{l}}$$

$$\dot f_1(l) = -\omega_0\sigma_2(l)$$

$$\dot\sigma_2(l) = \omega_0 f_1(l) - 4g\mu f_3(l)\cos \boldsymbol{k}\,.\,\boldsymbol{l} - 2gs_l(f\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{l}} + f^\dagger\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}.\boldsymbol{l}})$$

$$\dot f_3(l) = 4g\mu\cos \boldsymbol{k}\,.\,\boldsymbol{l}\,\sigma_2(l).$$

Below the critical temperature it would be incorrect to ignore the $f_3(l)$. After elimination of $f_3(l)$, $\sigma_2(l)$ and their derivatives from the set of equations, we find that solutions with time dependence $\exp(\mathrm{i}Et)$ require that

$$(E^2 - \omega_0^2\sec^2\theta_l)f_1(l) = -2g\omega_0 s_l(f\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{l}} + f^\dagger\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}.\boldsymbol{l}}).$$

The operators $f_1(l)$ may now be eliminated, leaving a pair of equations in f and $f^\dagger$ whose determinant implies the secular equation:

$$E^2-(\omega_k+S_\beta(E,0))^2+S_\beta^2(E,2\boldsymbol{k})=0 \tag{12}$$

where

$$S_\beta(E,\boldsymbol{q})\equiv 2g^2\omega_0\sum_l \frac{\cos\theta_l\cos\boldsymbol{q}\cdot\boldsymbol{l}\tanh(\frac{1}{2}\beta\omega_0\sec\theta_l)}{E^2-\omega_0^2\sec^2\theta_l}. \tag{13}$$

Upon setting $E=0$ in (12) it is found that for consistency β must satisfy (11).

In the case of two field modes with wavevectors $\boldsymbol{k}$ and $-\boldsymbol{k}$ in H, a parallel calculation gives the secular equation:

$$E^2-\omega_k^2=2\omega_k(S_\beta(E,0)\pm S_\beta(E,2\boldsymbol{k})) \tag{14}$$

which has the correct limit as $T\to 0$ (Thompson 1977). Both (12) and (14) predict a quasi-continuous band of localised modes which may be interpreted as position-dependent Stark-shifted levels. These shifts are temperature dependent and vanish with the coherent field as the temperature rises to its critical value. A simple graphical argument (Thompson 1977) gives these energies as $\omega_0\sec\theta_l$ approximately, a conclusion which generalises a result of Sadreev and Slavinskii (1976).

For long waves, (12) becomes

$$\begin{aligned}(E^2-\omega_k^2)(E^2-\omega_0^2)&=4g^2N\omega_0\omega_k\tanh(\tfrac{1}{2}\beta\omega_0) && T>T_c\\ (E^2-\omega_k^2)(E^2-\omega_0^2\eta^2)&=\omega_0^2\omega_k^2 && T<T_c\end{aligned} \tag{15}$$

with η satisfying

$$\eta=4g^2N\omega_0^{-1}\omega_k^{-1}\tanh(\tfrac{1}{2}\beta\omega_0\eta).$$

On resonance ($\omega_k=\omega_0$) the solutions at T_c are $E=0$, $\pm\sqrt{2}\omega_0$ showing that frequency doubling (Sadreev and Martynov 1976) is a feature of the RWA. Similarly, the temperature dependence of the soft-mode frequency near T_c is proportional to $|T-T_c|^{1/2}$. Figure 1 shows the solutions of (15) for resonance and $4g^2N/\omega_0\omega_k=1\cdot5$. Models in which the counter-rotating terms are given a fractional weighting x have been investigated in the long-wave limit. The dispersion relation is again a quadratic

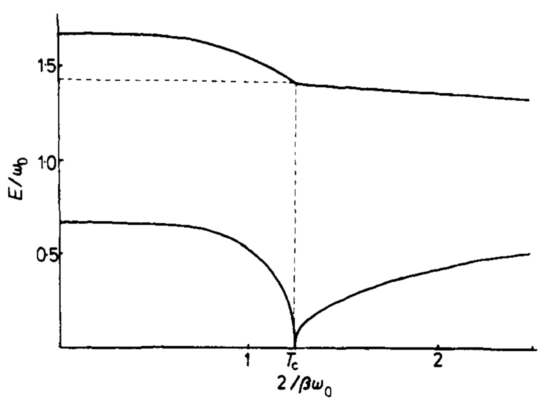

Figure 1. Temperature dependence of mode frequency.

in E^2 giving a half-power law for the soft mode near T_c when $x \neq 0$. Just above the critical temperature,

$$E^2 \approx c_1(T - T_c) + c_2(T - T_c)^2 + \ldots$$

but the coefficient c_1 tends to zero with x. This means that for small values of x, the half-power law region will be very close to T_c, the linear dependence becoming apparent farther from the transition. The extent of the half-power law region is an indicator of the importance of counter-rotating terms in these models of atom–field interactions near equilibrium.

I thank the referee for suggesting some improvements to the manuscript.

References

Cochran W 1973 *The Dynamics of Atoms in Crystals* (London: Edward Arnold) chap. 10
Gilmore R 1977 *Phys. Lett.* **60A** 387–8
Gilmore R and Bowden C M 1976 *Phys. Rev.* A **13** 1898–906
Sadreev A F and Martynov S N 1976 *Phys. Lett.* **59A** 183–4
Sadreev A F and Slavinskii A P 1976 *Fiz. Tverd. Tela* **18** 2169–74
Thompson B V 1977 *J. Phys. A: Math. Gen.* **10** 89–97